

# A BOCS THEORETIC CHARACTERIZATION OF GENDO-SYMMETRIC ALGEBRAS

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**ABSTRACT.** Gendo-symmetric algebras were recently introduced by Fang and König in [FanKoe]. An algebra is called gendo-symmetric in case it is isomorphic to the endomorphism ring of a generator over a finite dimensional symmetric algebra. We show that a finite dimensional algebra  $A$  over a field  $K$  is gendo-symmetric if and only if there is a bocs-structure on  $(A, D(A))$ , where  $D = \text{Hom}_K(-, K)$  is the natural duality. Assuming that  $A$  is gendo-symmetric, we show that the module category of the bocs  $(A, D(A))$  is isomorphic to the module category of the algebra  $eAe$ , when  $e$  is an idempotent such that  $eA$  is the unique minimal faithful projective-injective right  $A$ -module. We also prove some new results about gendo-symmetric algebras using the theory of bocses.

## INTRODUCTION

A boc is a generalization of the notion of coalgebra over a field. Bocses are also known under the name coring (see the book [BreWis]). A famous application of bocses has been the proof of the tame and wild dichotomy theorem by Drozd for finite dimensional algebras over an algebraically closed field (see [Dro] and the book [BSZ]). For any given boc  $(A, W)$  over a finite dimensional algebra, one can define a corresponding module category and analyze it. Given a finite dimensional algebra  $A$  over a field  $K$ , it is an interesting question whether for a given  $A$ -bimodule  $W$ , there exists a boc structure on  $(A, W)$ . The easiest example to consider is the case  $W = A$  and in this case the module category one gets is just the module category of the algebra  $A$ . Every finite dimensional algebra has a duality  $D = \text{Hom}_K(-, K)$  and so the next example of an  $A$ -bimodule to consider is perhaps  $W = D(A)$ . We will characterize all finite dimensional algebras  $A$  such that there is a boc structure on  $(A, D(A))$  and find a surprising connection to a recently introduced class of algebras generalizing symmetric algebras (see [FanKoe2]). Those algebras are called gendo-symmetric and are defined as endomorphism rings of generators of symmetric algebras. Alternatively these are the algebras  $A$ , where there exists an idempotent  $e$  such that  $eA$  is a minimal faithful injective-projective module and  $D(Ae) \cong eA$  as  $(eAe, A)$ -bimodules. Then  $eAe$  is the symmetric algebra such that  $A \cong \text{End}_{eAe}(M)$ , for an  $eAe$ -module  $M$  that is a generator of  $\text{mod-}eAe$ . Famous examples of non-symmetric gendo-symmetric algebras are Schur algebras  $S(n, r)$  with  $n \geq r$  and blocks of the Bernstein-Gelfand-Gelfand category  $\mathcal{O}$  of a complex semisimple Lie algebra (for a proof of this, using methods close to ours, see [KSX] and for applications see [FanKoe3]). The first section provides the necessary background on bocses and algebras with dominant dimension larger or equal 2. The second section proves our main theorem:

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2010 *Mathematics Subject Classification.* Primary 16G10, 16E10.

*Key words and phrases.* Representation theory of finite dimensional algebras, corings, dominant dimension.

**Theorem A**

(Theorem 2.2)

A finite dimensional algebra  $A$  is gendo-symmetric if and only if  $(A, D(A))$  has a bocs-structure.

We also provide some new structural results about gendo-symmetric algebras in this section. For example we show, using bocs-theoretic methods, that the tensor product over the field  $K$  of two gendo-symmetric algebras is again gendo-symmetric and we proof that  $\text{Hom}_{A^e}(D(A), A)$  is isomorphic to the center of  $A$ , where  $A^e$  denotes the enveloping algebra of  $A$ .

In the final section, we describe the module category  $\mathcal{B}$  of the bocs  $(A, D(A))$  in case  $A$  is gendo-symmetric. The following is our second main result:

**Theorem B**

(Theorem 3.3)

Let  $A$  be a gendo-symmetric algebra with minimal faithful projective-injective module  $eA$ . Then the module category of the bocs  $(A, D(A))$  is equivalent to  $eAe$ -mod as  $K$ -linear categories.

I thank Steffen König for useful comments and proofreading. I thank Julian Külshammer for providing me with an early copy of his article [Kue].

## 1. PRELIMINARIES

We collect here all needed definitions and lemmas to prove the main theorems. Let an algebra always be a finite dimensional algebra over a field  $K$  and a module over such an algebra is always a finite dimensional right module, unless otherwise stated.  $D = \text{Hom}_A(-, K)$  denotes the duality for a given finite dimensional algebra  $A$ .  $\text{mod} - A$  denotes the category of finite dimensional right  $A$ -modules and  $\text{proj}$  ( $\text{inj}$ ) denotes the subcategory of finitely generated projective (injective)  $A$ -modules. We note that we often omit the index in a tensor product, when we calculate with elements. We often identify  $A \otimes_A X \cong X$  for an  $A$ -module  $X$  without explicitly mentioning the natural isomorphism. The Nakayama functor  $\nu : \text{mod} - A \rightarrow \text{mod} - A$  is defined as  $D\text{Hom}_A(-, A)$  and is isomorphic to the functor  $(-) \otimes_A D(A)$ . The inverse Nakayama functor  $\nu^{-1} : \text{mod} - A \rightarrow \text{mod} - A$  is defined as  $\text{Hom}_{A^{op}}(-, A)D$  and is isomorphic to the functor  $\text{Hom}_A(D(A), -)$  (see [SkoYam] Chapter III section 5 for details). The Nakayama functors play a prominent role in the representation theory of finite dimensional algebras, since  $\nu : \text{proj} \rightarrow \text{inj}$  is an equivalence with inverse  $\nu^{-1}$ . For example they appear in the definition of the Auslander-Reiten translates  $\tau$  and  $\tau^{-1}$  (see [SkoYam] Chapter III. for the definitions):

**1.1. Proposition**

Let  $M$  be an  $A$ -module with a minimal injective presentation  $0 \rightarrow M \rightarrow I_0 \rightarrow I_1$ .

Then the following sequence is exact:

$$0 \rightarrow \nu^{-1}(M) \rightarrow \nu^{-1}(I_0) \rightarrow \nu^{-1}(I_1) \rightarrow \tau^{-1}(M) \rightarrow 0.$$

*Proof.* See [SkoYam], Chapter III. Proposition 5.3. (ii). □

The *dominant dimension*  $\text{domdim}(M)$  of a module  $M$  with a minimal injective resolution  $(I_i) : 0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow \dots$  is defined as:

$\text{domdim}(M) := \sup\{n | I_i \text{ is projective for } i = 0, 1, \dots, n\} + 1$ , if  $I_0$  is projective, and  $\text{domdim}(M) := 0$ , if  $I_0$  is not projective.

The dominant dimension of a finite dimensional algebra is defined as the dominant dimension of the regular module  $A_A$ . It is well-known that an algebra  $A$  has dominant dimension larger than or equal to 1 iff there is an idempotent  $e$  such that  $eA$  is a minimal faithful projective-injective module. The Morita-Tachikawa correspondence (see [Ta] for details) says that the algebras, which are endomorphism rings of

generator-cogenerators are exactly the algebras with dominant dimension at least 2. The full subcategory of modules of dominant dimension at least  $i \geq 1$  is denoted by  $Dom_i$ .  $A$  is called a *Morita algebra* iff it has dominant dimension larger than or equal to 2 and  $D(Ae) \cong eA$  as  $A$ -right modules. This is equivalent to  $A$  being isomorphic to  $End_B(M)$ , where  $B$  is a selfinjective algebra and  $M$  a generator of  $\text{mod-}B$  (see [KerYam]).  $A$  is called a *gendo-symmetric algebra* iff it has dominant dimension larger than or equal to 2 and  $D(Ae) \cong eA$  as  $(eAe, A)$ -bimodules iff it has dominant dimension larger than or equal to 2 and  $D(eA) \cong Ae$  as  $(A, eAe)$ -bimodules. This is equivalent to  $A$  being isomorphic to  $End_B(M)$ , where  $B$  is a symmetric algebra and  $M$  a generator of  $\text{mod-}B$  and in this case  $B = eAe$  (see [FanKoe]).

### 1.2. Proposition

Let  $A$  be a gendo-symmetric algebra and  $M$  an  $A$ -module. Then  $M$  has dominant dimension larger or equal to two iff  $\nu^{-1}(M) \cong M$ .

*Proof.* See [FanKoe2], proposition 3.3.  $\square$

The following result gives a formula for the dominant dimension of Morita algebras:

### 1.3. Proposition

Let  $A$  be a Morita algebra with minimal faithful projective-injective module  $eA$  and  $M$  an  $A$ -module. Then  $\text{domdim}(M) = \inf\{i \geq 0 \mid \text{Ext}^i(A/AeA, M) \neq 0\}$ . Especially,  $\text{Hom}_A(A/AeA, A) = 0$  for every Morita algebra, since they always have dominant dimension at least 2.

*Proof.* This is a special case of [APT], Proposition 2.6.  $\square$

The following lemma gives another characterization of gendo-symmetric algebras, which is used in the proof of the main theorem.

### 1.4. Lemma

Let  $A$  be a finite dimensional algebra. Then  $A$  is a gendo-symmetric algebra iff  $D(A) \otimes_A D(A) \cong D(A)$  as  $A$ -bimodules. Assume  $eA$  is the minimal faithful projective-injective module. In case  $A$  is gendo-symmetric,  $D(A) \cong Ae \otimes_{eAe} eA$  as  $A$ -bimodules.

*Proof.* See [FanKoe2] Theorem 3.2. and [FanKoe] in the construction of the multiplication following Definition 2.3.  $\square$

### 1.5. Lemma

An  $A$ -module  $P$  is projective iff there are elements  $p_1, p_2, \dots, p_n \in P$  and elements  $\pi_1, \pi_2, \dots, \pi_n \in \text{Hom}_A(P, A)$  such that the following condition holds:

$$x = \sum_{i=1}^n p_i \pi_i(x) \text{ for every } x \in P.$$

We then call the  $p_1, \dots, p_n$  a *probasis* and  $\pi_1, \dots, \pi_n$  a *dual probasis* of  $P$ .

*Proof.* See [Rot] Proposition 3.10.  $\square$

### 1.6. Example

Let  $P = eA$ , for an idempotent  $e$ . Then a probasis is given by  $p_1 = e$  and the dual probasis is given by  $\pi_1 = l_e \in \text{Hom}_A(eA, A)$ , which is left multiplication by  $e$ .  $l_e$  can be identified with  $e$  under the  $(A, eAe)$ -bimodule isomorphism  $Ae \cong \text{Hom}_A(eA, A)$ .

### 1.7. Proposition

1.  $\text{Hom}_A(D(A), A)$  is a faithful right  $A$ -module iff there is an idempotent  $e$ , such that  $eA$  and  $Ae$  are faithful and injective.
2. Let  $A$  be an algebra with  $\text{Hom}_A(D(A), A) \cong A$  as right  $A$ -modules, then  $A$  is a Morita algebra.

*Proof.* 1. See [KerYam], Theorem 1.

2. See [KerYam], Theorem 3.  $\square$

### 1.8. Lemma

Let  $Y$  and  $Z$  be  $A$ -bimodules. Then the following is an isomorphism of  $A$ -bimodules:

$$\text{Hom}_A(Y, D(Z)) \cong D(Y \otimes_A Z).$$

*Proof.* See [ASS] Appendix 4, Proposition 4.11.  $\square$

### 1.9. Definition

Let  $A$  be a finite dimensional algebra and  $W$  an  $A$ -bimodule and let  $c_l : W \rightarrow A \otimes_A W$  and  $c_r : W \rightarrow W \otimes_A A$  be the canonical isomorphisms. Then the tuple  $\mathcal{B} := (A, W)$  is called a *bocs* (see [Kue]) or the module  $W$  is called an  $A$ -coring (see [BreWis]) if there are  $A$ -bimodule maps  $\mu : W \rightarrow W \otimes_A W$  (the comultiplication) and  $\epsilon : W \rightarrow A$  (the counit) with the following properties:

$(1_W \otimes_A \epsilon)\mu = c_l$ ,  $(\epsilon \otimes_A 1_W)\mu = c_r$  and  $(\mu \otimes_A 1_W)\mu = (1_W \otimes_A \mu)\mu$ . We often say for short that  $W$  is a boc, if  $A$  (and  $\mu$  and  $\epsilon$ ) are clear from the context. The category of the finite dimensional boc modules is defined as follows:

Objects are the finite dimensional right  $A$ -modules.

Homomorphism spaces are  $\text{Hom}_{\mathcal{B}}(M, N) := \text{Hom}_A(M, \text{Hom}_A(W, N))$  with the following composition  $*$  and units:

Let  $g : M \rightarrow \text{Hom}_A(W, N) \in \text{Hom}_{\mathcal{B}}(M, N)$  and  $f : L \rightarrow \text{Hom}_A(W, M) \in \text{Hom}_{\mathcal{B}}(L, M)$ . Then  $g * f := \text{Hom}_A(\mu, N)\psi\text{Hom}_A(W, g)f$ , where  $\psi$  is the adjunction isomorphism  $\text{Hom}_A(W, \text{Hom}_A(W, N)) \rightarrow \text{Hom}_A(W \otimes_A W, N)$ . The units  $1_M \in \text{Hom}_{\mathcal{B}}(M, M)$  are defined as follows:  $1_M := \text{Hom}_A(\epsilon, M)\xi$ , where  $\xi : M \rightarrow \text{Hom}_A(A, M)$  is the canonical isomorphism. Note that the module category of a boc is  $K$ -linear. We refer to [Kue] for other equivalent descriptions of the boc module category and more information.

### 1.10. Examples

1.  $(A, A)$  is always a boc with the obvious multiplication and comultiplication. The next natural bimodule to look for a boc-structure is  $D(A)$ . We will see that  $(A, D(A))$  is not a boc for arbitrary finite dimensional algebras.

2. The next example can be found in 17.6. in [BreWis], to which we refer for more details. Let  $P$  be a  $(B, A)$ -bimodule for two finite dimensional algebras  $B$  and  $A$  such that  $P$  is projective as a right  $A$ -module and let  $P^* := \text{Hom}(P, A)$ , which is then a  $(A, B)$  bimodule. Let  $p_1, p_2, \dots, p_n$  be a probasis for  $P$  and  $\pi_1, \pi_2, \dots, \pi_n$  a dual probasis of the projective  $A$ -module  $P$ . Denote the  $A$ -bimodule  $P^* \otimes_B P$  by  $W$  and define the comultiplication  $\mu : W \rightarrow W \otimes_A W$  as follows: Let  $f \in P^*$  and  $p \in P$ , then  $\mu(f \otimes p) = \sum_{i=1}^n (f \otimes p_i) \otimes (\pi_i \otimes p)$ . Define the counit  $\epsilon : W \rightarrow A$  as follows:  $\epsilon(f \otimes p) = f(p)$ . Now specialise to  $P = eA$ , for an idempotent  $e$  and identify  $\text{Hom}_A(eA, A) = Ae$ . Then  $\mu(ae \otimes eb) = (ae \otimes e) \otimes (e \otimes eb)$  and  $\epsilon(ae \otimes eb) = aeb$ . We will use this special case in the next section to show that  $(A, D(A))$  is always a boc for a gendo-symmetric algebra.

3. Let  $(A_1, W_1)$  and  $(A_2, W_2)$  be bocses, then  $(A_1 \otimes_K A_2, W_1 \otimes_K W_2)$  is again a boc. See [BreWis] 24.1. for a proof.

## 2. CHARACTERIZATION OF GENDO-SYMMETRIC ALGEBRAS

The following lemma, will be important for proving the main theorem.

### 2.1. Lemma

Assume that  $\text{Hom}_A(D(A), A) \cong A \oplus X$  as right  $A$ -modules for some right  $A$ -module  $X$ , then  $\text{domdim}(A) \geq 2$  and  $X = 0$ .

*Proof.* By assumption  $\text{Hom}_A(D(A), A)$  is faithful and so there is an idempotent  $e$  with  $eA$  and  $Ae$  faithful and injective by 1.7 1., which implies that  $A$  has dominant dimension at least 1. Choose  $e$  minimal such that those properties hold. Now look at the minimal injective presentation  $0 \rightarrow A \rightarrow I_0 \rightarrow I_1$  of  $A$  and note that  $I_0 \in \text{add}(eA)$ . Using 1.1, there is the following exact sequence:  $0 \rightarrow \nu^{-1}(A) \rightarrow \nu^{-1}(I_0) \rightarrow \nu^{-1}(I_1) \rightarrow \tau^{-1}(A) \rightarrow 0$ . But  $\nu^{-1}(A) \cong \text{Hom}_A(D(A), A) \cong A \oplus X$  and so there is the embedding:  $0 \rightarrow A \oplus X \rightarrow \nu^{-1}(I_0)$ . Note that  $\nu^{-1}(I_0) \in \text{add}(eA)$  is the injective hull of  $A \oplus X$ , since  $\nu^{-1} : \text{inj} \rightarrow \text{proj}$  is an equivalence and  $eA$  is the minimal faithful projective injective module. Thus  $\nu^{-1}(I_0)$  has the same number of indecomposable direct summands as  $I_0$ . Therefore  $\text{soc}(X) = 0$  and so  $X = 0$ , since every indecomposable summand of the socle of the module provides an indecomposable direct summand of the injective hull of that module. Thus  $\text{Hom}_A(D(A), A) \cong A$  and  $A$  is a Morita algebra by 1.7 2. and so  $A$  has dominant dimension at least 2.  $\square$

We now give a bocs-theoretic characterization of gendo-symmetric algebras.

## 2.2. Theorem

Let  $A$  be a finite dimensional algebra. Then the following are equivalent:

1.  $A$  is gendo-symmetric.
2. There is a comultiplication and counit such that  $\mathcal{B} = (A, D(A))$  is a bocs.

*Proof.* We first show that 1. implies 2.:

Assume that  $A$  is gendo-symmetric with minimal faithful projective-injective module  $eA$ . Set  $P := eA$  and apply the second example in 1.10, with  $B := eAe$ , to see that  $\mathcal{B} := (A, Ae \otimes_{eAe} eA)$  has the structure of a bocs. Now note that by 1.4  $D(A) \cong Ae \otimes_{eAe} eA$  as  $A$ -bimodules and one can use this to get a bocs structure for  $(A, D(A))$ .

Now we show that 2. implies 1.:

Assume that  $(A, D(A))$  is a bocs with comultiplication  $\mu$  and counit  $\epsilon$ . Note first that the comultiplication  $\mu$  always has to be injective because in the identity  $(\epsilon \otimes_A 1_W)\mu = c_r$  appearing the definition of a bocs,  $c_r$  is an isomorphism. So there is a injection  $\mu : D(A) \rightarrow D(A) \otimes_A D(A)$  which gives a surjection  $D(\mu) : D(D(A) \otimes_A D(A)) \rightarrow A$ . Now using 1.8 we see that  $D(D(A) \otimes_A D(A)) \cong \text{Hom}_A(D(A), A)$  as  $A$ -bimodules.

Since  $A$  is projective,  $D(\mu)$  is split and  $\text{Hom}_A(D(A), A) \cong A \oplus X$  for some  $A$ -right module  $X$ . By 2.1, this implies  $\text{Hom}_A(D(A), A) \cong A$  and comparing dimensions,  $D(\mu)$  and thus also  $\mu$  have to be isomorphisms. By 1.4,  $A$  is gendo-symmetric.  $\square$

## 2.3. Corollary

Let  $A$  be a finite dimensional algebra. Then the following two conditions are equivalent:

1.  $A$  is gendo-symmetric.
2.  $\nu$  is a comonad.

*Proof.* In [BreWis] 18.28. it is proven that an  $A$ -bimodule  $W$  is a bocs iff the functor  $(-) \otimes_A W$  is a comonad. Applying this with  $W = D(A)$  and using the previous theorem, the corollary follows.  $\square$

## 2.4. Remark

Theorem 2.2 also shows that the comultiplication of the bocs  $(A, D(A))$  is always an  $A$ -bimodule isomorphism for a gendo-symmetric algebra  $A$ . In [FanKoe], section 2.2., it is noted that such an isomorphism is unique up to multiples of invertible central elements in  $A$ . Thus the comultiplication of the bocs is also unique in that sense.

The following proposition gives an application:

### 2.5. Proposition

Let  $A$  and  $B$  be gendo-symmetric  $K$ -algebras. Then  $A \otimes_K B$  is again a gendo-symmetric  $K$ -algebra. In particular, let  $F$  be a field extension of  $K$  and  $A$  a gendo-symmetric  $K$ -algebra. Then  $A \otimes_K F$  is again gendo-symmetric.

*Proof.* Let  $A$  and  $B$  two gendo-symmetric algebras. Then  $\mathcal{B}_1 = (A, D(A))$  and  $\mathcal{B}_2 = (B, D(B))$  are bocses. By example 3 of 1.10 also the tensor product of  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are bocses, it is the bocs  $\mathcal{C} = (A \otimes_K B, D(A) \otimes_K D(B))$ . Recall the well known formula  $(D(A) \otimes_K D(B)) \cong D(A \otimes_K B)$ , which can be found as exercise 12. of chapter II. in [SkoYam]. Using this isomorphism one can find a bocs structure on  $(A \otimes_K B, D(A \otimes_K B))$  using the bocs structure on  $\mathcal{C}$ . Thus by our bocs-theoretic characterization of gendo-symmetric algebras, also  $A \otimes_K B$  is gendo-symmetric. The second part follows since every field is a symmetric and thus gendo-symmetric algebra.  $\square$

Let  $A^e := A^{op} \otimes_K A$  denote the enveloping algebra of a given algebra  $A$ . The following proposition can be found in [BreWis], 17.8.

### 2.6. Proposition

Let  $(A, W)$  be a bocs and  $c \in W$  with  $\mu(c) = \sum_{i=1}^n c_{1,i} \otimes c_{2,i}$ .

1.  $\text{Hom}_A(W, A)$  has a ring structure with unit  $\epsilon$  and product  $*^r$ , given as follows for  $f, g \in \text{Hom}_A(W, A)$ :

$$f *^r g = g(f \otimes_A \text{id}_W) \mu.$$

There is a ring anti-morphism  $\zeta : A \rightarrow \text{Hom}_A(W, A)$ , given by  $\zeta(a) = \epsilon(a(-))$ .

2.  $\text{Hom}_{A^e}(W, A)$  has a ring structure with unit  $\epsilon$  and multiplication  $*$  given as follows for  $f, g \in \text{Hom}_{A^e}(W, A)$ :

$$f * g(c) = \sum_{i=1}^n f(c_{1,i}) g(c_{2,i}).$$

We now describe the ring structures on  $\text{Hom}_{A^e}(D(A), A)$  and  $\text{Hom}_A(D(A), A)$ .

### 2.7. Proposition

Let  $A$  be gendo-symmetric.

1.  $\zeta$ , as defined in the previous proposition, is a ring anti-isomorphism  $\zeta : A \rightarrow \text{Hom}_A(D(A), A)$ .

2. With the ring structure on  $\text{Hom}_{A^e}(D(A), A)$  as defined in the previous proposition,  $\text{Hom}_{A^e}(D(A), A)$  is isomorphic to the center  $Z(A)$  of  $A$ .

*Proof.* We use the isomorphism of  $A$ -bimodule  $D(A) \cong Ae \otimes_{eAe} eA$ .

1. Since  $A$  and  $\text{Hom}_A(D(A), A)$  have the same  $K$ -dimension, the only thing left to show is that  $\zeta$  is injective. So assume that  $\zeta(a) = \epsilon(a(-)) = 0$ , for some  $a \in A$ . This is equivalent to  $\epsilon(ax) = 0$  for every  $x = ce \otimes ed \in Ae \otimes eA$ . Now  $\epsilon(a(ce \otimes ed)) = \epsilon(ace \otimes ed) = aced$ . Thus, since  $c, d$  were arbitrary,  $aAeA = 0$ . This means that  $a$  is in the left annihilator  $L(AeA)$  of the two-sided ideal  $AeA$ . But  $L(AeA) = 0$ , since  $\text{Hom}_A(A/AeA, A) = 0$ , by 1.3 and thus  $a = 0$ . Therefore  $\zeta$  is injective.

2. Define  $\psi : \text{Hom}_{A^e}(D(A), A) \rightarrow Z(eAe)$  by  $\psi(f) = f(e \otimes e)$ , for

$f \in \text{Hom}_{A^e}(D(A), A)$ . First, we show that this is well-defined, that is  $f(e \otimes e)$  is really in the center of  $Z(eAe)$ . Let  $x \in eAe$ . Then  $xf(e \otimes e) = f(xe \otimes e) = f(e \otimes ex) = f(e \otimes e)x$  and therefore  $f(e \otimes e) \in Z(eAe)$ . Clearly,  $\psi$  is  $K$ -linear. Now we show that the map is injective: Assume  $\psi(f) = 0$ , which is equivalent to  $f(e \otimes e) = 0$ . Then for any  $a, b \in A : f(ae \otimes eb) = 0$ , and thus  $f = 0$ .

Now we show that  $\psi$  is surjective. Let  $z \in Z(eAe)$  be given. Then define a map

$f_z \in \text{Hom}_{A^e}(D(A), A)$  by  $f_z(ae \otimes eb) = zaeb$ . Then, since  $z$  is in the center of  $eAe$ ,  $f$  is  $A$ -bilinear and obviously  $\psi(f_z) = f_z(e \otimes e) = ze = z$ .  $\psi$  also preserves the unit and multiplication:

$\psi(\epsilon) = \epsilon(e \otimes e) = e^2 = e$  and for two given  $f, g \in \text{Hom}_{A^e}(D(A), A)$ :  $\phi(f * g) = (f * g)(e \otimes e) = (f * g)(e \otimes e) = f(e \otimes e)g(e \otimes e)$ , by the definition of  $*$ . To finish the proof, we use the result from [FanKoe], Lemma 2.2., that the map  $\phi : Z(A) \rightarrow Z(eAe)$ ,  $\phi(z) = eze$  is a ring isomorphism in case  $A$  is gendo-symmetric.  $\square$

### 3. DESCRIPTION OF THE MODULE CATEGORY OF THE BOCS $(A, D(A))$ FOR A GENDO-SYMMETRIC ALGEBRA

Let  $A$  be a gendo-symmetric algebra. In this section we describe the module category of the boc  $\mathcal{B} = (A, D(A))$  as a  $K$ -linear category. We will use the  $A$ -bimodule isomorphism  $Ae \otimes_{eAe} eA \cong D(A)$  often without mentioning. Let  $M$  be an arbitrary  $A$ -module. Define for a given  $M$  the map  $I_M : M \rightarrow \text{Hom}_A(D(A), M)$  by  $I_M(m) = u_m$  for any  $m \in M$ , where  $u_m : D(A) \rightarrow M$  is the map  $u_m(ae \otimes eb) = maeb$  for any  $a, b \in A$ . Before we get into explicit calculation, let us recall how  $*$  is defined in this special case. Let  $f \in \text{Hom}_{\mathcal{B}}(L, M)$  and  $g \in \text{Hom}_{\mathcal{B}}(M, N)$ , then for  $l \in L$  and  $a, b \in A$ :  $(g * f)(l)(ae \otimes eb) = g(f(l)(ae \otimes e))(e \otimes eb)$ .

#### 3.1. Proposition

1.  $I_M$  is well defined.
2.  $I_M$  is injective, iff  $M$  has dominant dimension larger or equal 1.
3.  $I_M$  is bijective, iff  $M$  has dominant dimension larger or equal 2.

*Proof.* 1. We have to show two things: First,  $u_m$  is  $A$ -linear for any  $m \in M$ :  $u_m((ae \otimes eb)c) = u_m(ae \otimes ebc) = maebc = (maeb)c = u_m(ae \otimes eb)c$ . Second,  $I_M$  is also  $A$ -linear:  $I_M(mc)(ae \otimes eb) = u_{mc}(ae \otimes eb) = mcaeb = u_m(cae \otimes eb) = (u_m c)(ae \otimes eb) = (I_M(m)c)(ae \otimes eb)$ .

2.  $I_M$  is injective iff  $(m = 0 \Leftrightarrow u_m = 0)$ . Now  $u_m = 0$  is equivalent to  $maeb = 0$  for any  $a, b \in A$ . This is equivalent to the condition that the two-sided ideal  $AeA$  annihilates  $m$ . Thus there is a nonzero  $m$  with  $u_m = 0$  iff  $\text{Hom}_A(A/AeA, M) \neq 0$  iff  $M$  has dominant dimension zero by 1.3.

3. By 1.2  $M$  has dominant dimension larger or equal two iff  $M \cong \nu^{-1}(M)$ .

Thus 3. follows by 2. since an injective map between modules of the same dimension is a bijective map.  $\square$

#### 3.2. Lemma

For any module  $M$ , there is an isomorphism

$\text{Hom}_A(\mu, M)\psi : \text{Hom}_A(D(A), \text{Hom}_A(D(A), M)) \rightarrow \text{Hom}(D(A), M)$  and thus  $\nu^{-1}(M) \cong \nu^{-2}(M)$ . It follows that every module of the form  $\nu^{-1}(M)$  has dominant dimension at least two.

*Proof.* The result follows, since  $\psi$  is the canonical isomorphism

$\psi : \text{Hom}_A(D(A), \text{Hom}_A(D(A), M)) \rightarrow \text{Hom}_A(D(A) \otimes_A D(A), M)$  and since  $\mu$  is an isomorphism also  $\text{Hom}_A(\mu, M)$  is an isomorphism. That  $\nu^{-1}(M)$  has dominant dimension at least two, follows now from 1.2.  $\square$

We define a functor  $\phi : \text{mod-}A \rightarrow \text{mod-}\mathcal{B}$  by  $\phi(M) = M$  and  $\phi(f) = I_N f$  for an  $A$ -homomorphism  $f : M \rightarrow N$ .  $\phi$  is obviously  $K$ -linear. The next result shows that it really is a functor and calculates its kernel on objects.

#### 3.3. Theorem

1.  $\phi$  is a  $K$ -linear functor.
2.  $\phi(M) = 0$  iff the two-sided ideal  $AeA$  annihilates  $M$ , that is  $M$  is an  $A/AeA$ -module. All modules  $M$  that are annihilated by  $AeA$  have dominant dimension

zero.

3. By restricting  $\phi$  to  $Dom_2$ , one gets an equivalence of  $K$ -linear categories  $Dom_2 \rightarrow Dom_2^{\mathcal{B}}$ , where  $Dom_2^{\mathcal{B}}$  denotes the full subcategory of  $mod - \mathcal{B}$  having objects all modules of dominant dimension at least 2.

4. Any module  $A$ -module  $M$  is isomorphic to  $\nu^{-1}(M)$  in  $\mathcal{B}$ -mod and thus  $\mathcal{B}$ -mod is equivalent to  $Dom_2$  as  $K$ -linear categories, which is equivalent to the module category  $mod-eAe$ .

*Proof.* 1. It was noted above that  $\phi$  is  $K$ -linear. We have to show  $\phi(id_M) = Hom(\epsilon, M)\zeta$ , where  $\zeta : M \rightarrow Hom_A(A, M)$  is the canonical isomorphism, and  $\phi(g \circ f) = I_N(g) * I_M(f)$ , where  $f : L \rightarrow M$  and  $g : M \rightarrow N$  are  $A$ -module homomorphisms. To show the first equality  $\phi(id_M) = Hom(\epsilon, M)\zeta$ , just note that  $Hom(\epsilon, M)\zeta(m)(ae \otimes eb) = l_m(\epsilon(ae \otimes eb)) = maeb = I_M(m)(ae \otimes eb)$ , where  $l_m : A \rightarrow M$  is left multiplication by  $m$ .

Next we show the above equality  $\phi(g \circ f) = I_N(g) * I_M(f)$ :

Let  $l \in L$  and  $a, b \in A$ . First, we calculate  $\phi(g \circ f)(l)(ae \otimes eb) = g(f(l))aeb$ .

Second,  $I_N(g) * I_M(f)(l)(ae \otimes eb) = I_N(g)(I_M(f)(l)(ae \otimes e))(e \otimes eb) =$

$I_N(g)(u_{f(l)}(ae \otimes e))(e \otimes eb) = I_N(g)(f(l)(ae))(e \otimes eb) = g(f(l))aeb$ .

Thus  $\phi(g \circ f) = I_N(g) * I_M(f)$  is shown.

2. A module  $M$  is zero in the  $K$ -category  $mod-\mathcal{B}$  iff its endomorphism ring  $End_{\mathcal{B}}(M)$  is zero iff the identity of  $End_{\mathcal{B}}(M)$  is zero. Thus  $M$  is zero in  $mod-\mathcal{B}$  iff  $I_M(m) = 0$  for every  $m \in M$ . But  $I_M(m) = 0$  iff  $mAeA = 0$  and so  $\phi(M) = 0$  iff  $MAeA = 0$ . To see that such an  $M$  must have dominant dimension zero, note that  $AeA$  annihilates no element of  $M$  iff  $M$  has dominant dimension larger or equal 1 by 1.3.

3. Restricting  $\phi$  to  $Dom_2$ ,  $\phi$  is obviously still dense by the definition of  $Dom_2^{\mathcal{B}}$ . Now recall that by the previous proposition a module  $M$  has dominant dimension at least two iff  $I_M$  is an isomorphism. Let now  $h \in Hom_{\mathcal{B}}(M, N)$  be given with  $M, N \in Dom_2^{\mathcal{B}}$ . Then  $\phi(I_N^{-1}h) = I_N(I_N^{-1}h) = h$  and  $\phi$  is full. Assume  $\phi(h) = I_Nh = 0$ , then  $h = 0$ , since  $I_N$  is an isomorphism, and so  $\phi$  is faithful.

4. Define  $f \in Hom_{\mathcal{B}}(M, \nu^{-1}(M))$  as  $f = (Hom_A(\mu, M)\psi)^{-1}I_M$  and  $g \in Hom_{\mathcal{B}}(\nu^{-1}(M), M)$  as  $g = id_{\nu^{-1}(M)}$ . We show that  $f * g = I_{\nu^{-1}(M)}$  and  $g * f = I_M$ , which by 1. are the identities of  $Hom_{\mathcal{B}}(\nu^{-1}(M), \nu^{-1}(M))$  and  $Hom_{\mathcal{B}}(M, M)$ . This shows that any module  $M$  is isomorphic to  $\nu^{-1}(M)$  in  $\mathcal{B}$ -mod.

Let  $m \in M$  and  $a, b \in A$ .

Then  $(g * f)(m)(ae \otimes eb) = g(f(m)(ae \otimes e))(e \otimes eb) = ((Hom_A(\mu, M)\psi)^{-1}I_M(m))(ae \otimes e)(e \otimes eb) = maeb = I_M(m)(ae \otimes eb)$ , where we used that  $g$  is the identity on  $\nu^{-1}(M)$ . Next we show that  $f * g = I_{\nu^{-1}(M)}$ : Let  $l \in \nu^{-1}(M) = Hom_A(D(A), M)$ .

First, note that by definition  $I_{\nu^{-1}(M)}(l)(ae \otimes eb)(a'e \otimes eb') = (laeb)(a'e \otimes eb') = l(aeba'e \otimes eb')$ . Next  $(f * g)(l)(ae \otimes eb)(a'e \otimes eb') = f(g(l)(ae \otimes eb)(a'e \otimes eb')) = f(l(ae \otimes eb)(a'e \otimes eb')) = (Hom_A(\mu, M)\psi)^{-1}I_M(l(ae \otimes eb)(a'e \otimes eb')) = l(ae \otimes eba'eb') = l(aeba'e \otimes eb')$ , where we used in the last step that we tensor over  $eAe$ .

Now we use 3.2, to show that every module of the form  $\nu^{-1}(M)$  has dominant dimension at least two. Since every module  $M$  is isomorphic to  $\nu^{-1}(M)$ ,  $\mathcal{B} - mod$  is isomorphic to  $Dom_2^{\mathcal{B}}$ , which is isomorphic to  $Dom_2$  by 3. Now recall that there is an equivalence of categories  $mod-eAe \rightarrow Dom_2$  (this is a special case of [APT] Lemma 3.1.). Combining all those equivalences, we get that  $\mathcal{B} - mod$  is equivalent to the module category  $mod-eAe$ .  $\square$

### 3.4. Corollary

In case an  $A$ -module  $M$  has dominant dimension larger or equal 2, the map



$\text{Hom}_A(M, I_M) : \text{End}_A(M) \rightarrow \text{End}_B(M)$  is a  $K$ -algebra isomorphism. In particular  $A \cong \text{End}_A(A) \cong \text{End}_B(A)$ , since  $A$  has dominant dimension at least 2.

*Proof.* This follows since  $I_M$  is an isomorphism, in case  $M$  has dominant dimension at least two by 3.1 3. □

### 3.5. Example

Let  $n \geq 2$  and  $A := K[x]/(x^n)$  and  $J$  the Jacobson radical of  $A$ . Let  $M := A \oplus \bigoplus_{k=1}^{n-1} J^k$  and  $B := \text{End}_A(M)$ . Then  $B$  is the Auslander algebra of  $A$  and  $B$  has  $n$  simple modules. The idempotent  $e$  is in this case primitive and corresponds to the unique indecomposable projective-injective module  $\text{Hom}_A(M, A)$ . By the previous theorem, the kernel of  $\phi$  is isomorphic to the module category  $\text{mod} - (A/AeA)$ . Here  $A/AeA$  is isomorphic to the preprojective algebra of type  $A_{n-1}$  by [DR] chapter 7.

We describe the boc module category  $\mathcal{B}\text{-mod}$  of  $(B, D(B))$  for  $n = 2$  explicitly. In this case  $B$  is isomorphic to the Nakayama algebra with Kupisch series [2, 3]. Then  $B$  has five indecomposable modules. Let  $e_0$  be the primitive idempotent corresponding to the indecomposable projective module with dimension two and  $e_1$  the primitive idempotent corresponding to the indecomposable projective module with dimension three. Then  $e_1A$  is the unique minimal faithful indecomposable projective-injective module. Let  $S_i$  denote the simple  $B$ -modules. The only indecomposable module annihilated by  $Be_1B$  is  $S_0$ , which is therefore isomorphic to zero in the boc module category. The two indecomposable projective modules  $P_0 = e_0B$  and  $P_1 = e_1B$  have dominant dimension at least two and thus are not isomorphic. The only indecomposable module of dominant dimension 1 is  $S_1$  and the only indecomposable module of dominant dimension zero, which is not isomorphic to zero in  $\mathcal{B}\text{-mod}$ , is  $D(Be_0)$ . Now let  $X = S_1$  or  $X = D(Be_0)$ , then  $\nu^{-1}(X) = \text{Hom}_B(D(B), X) \cong e_0B$ . Thus in  $\mathcal{B}\text{-mod}$   $S_1 \cong e_0B \cong D(Be_0)$  and  $e_1B$  are up to isomorphism the unique indecomposable objects.

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